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COVARIANT INTEGRALS IN MECHANICS

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Using a newly introduced concept of an integral in the tensor calculus [1], we obtain novel covariant integrals of equations of motion of a system for generalized forces of specific structure. We consider the problem of full covariance of the equations of motion in the configurational space, and obtain a new covariant form of the equations of motion of a system in curvilinear coordinates. The covariant integrability of these equations leads to reformulation of a fundamental axiom of dynamics.

1. We know that the second order Lagrange equations of motion for a holonomic system

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\alpha}} - \frac{\partial T}{\partial q^{\alpha}} = a_{\alpha\beta}\ddot{q}^{\beta} - \Gamma_{\alpha\beta,\gamma}\dot{q}^{\beta}\dot{q}^{\gamma} = Q_{\alpha} \tag{1.1}$$

can yield, under certain conditions, integrals of energy and impulse, or cyclic integrals. Other general integrals of the Lagrange equations (1.1) cannot be obtained even when the generalized forces are equal to zero.

We find that we can use the absolute tensor integral [1] to integrate the system of equations in question in its general form not only for the case when $Q_{\alpha} = 0$, but also for a wider class of generalized forces $Q_{\alpha} = Q_{\alpha}(q^1, \ldots, q^n, \dot{q}^1 \ldots, \dot{q}^n; t)$.

In fact, let us write the differential equations (1.1) in the form

$$\frac{D}{dt}\left(\frac{\partial T}{\partial \dot{q}^{\alpha}}\right) = \frac{D}{dt}(a_{\alpha\beta}\dot{q}^{\beta}) = Q_{\alpha} \tag{1.2}$$

Here D / dt denotes the operator of the absolute time derivative and $p_{\alpha} = a_{\alpha\beta}\dot{q}^{\beta}$ is the generalized impulse of the system in the V_n-configurational space. (The latter relation is also valid for a rheonomous system in an extended $(n + 1)$-dimensional space V_{n+1}, but here we shall only consider a scleronomous system).

We shall show that Eqs. (1.2) admit, under certain conditions, covariant integrals in the case of generalized forces of the form

$$Q_{\alpha} = R_{\alpha} + S_{\alpha\beta}F^{\beta} \quad (\alpha, \beta = 1, 2, \ldots, n)$$
$$R_{\alpha} = R_{\alpha}(q^1, q^2, \ldots, q^n), \quad S_{\alpha\beta} = S_{\alpha\beta}(q^1, q^2, \ldots, q^n) \tag{1.3}$$

where R_{α} are the coordinates of the parallel-translatable vector, $S_{\alpha\beta}$ is the covariantly constant tensor and $F^{\beta} = F^{\beta}(t)$ represent the integrable functions of time t.

Substituting (1.3) into (1.2), we obtain a system of initial covariant equations of motion in the form

$$D(\partial T / \partial \dot{q}^{\alpha}) = (R_{\alpha} + S_{\alpha\beta}F^{\beta})\, Dt \tag{1.4}$$

since the absolute differential Dt of the scalar t is equal to the differential dt of this same scalar.

In [1] it is shown that the absolute tensor integral

$$\hat{\int} DU^{\beta\cdots}_{\alpha\ldots}(X) = U^{\beta\cdots}_{\alpha\ldots}(X) - A^{\beta\cdots}_{\alpha\ldots}(X, M)$$

where $A^{\beta\cdots}_{\alpha\ldots}(X, M)$ is the covariantly constant tensor which is obtained from the initial condition by parallel translation along the trajectories, as is shown in [2].

Let us perform the absolute tensor integration on both sides of Eq. (1.4)

$$\hat{\int} D\left(\frac{\partial T}{\partial \dot{q}^{\alpha}}\right) = \hat{\int} (R_\alpha + S_{\alpha\beta} F^{\beta}) Dt$$

Taking into account the singularities of the absolute tensor integral, we obtain

$$\hat{\int} D\left(\frac{\partial T}{\partial \dot{q}^{\alpha}}\right) = \frac{\partial T}{\partial \dot{q}^{\alpha}} - A^1_\alpha$$

From the assumption that $R_\alpha = R_\alpha(q^1, \ldots, q^n)$ and $S_{\alpha\beta} = S_{\alpha\beta}(q^1, \ldots, q^n)$ are parallel-translatable tensors [3], we have

$$\hat{\int} R_\alpha Dt = tR_\alpha(q^1, \ldots, q^n) - A_\alpha^2$$

$$\hat{\int} S_{\alpha\beta} F^\beta Dt = S_{\alpha\beta} \hat{\int} F^\beta(t)\, Dt - A_\alpha^3 =$$

$$S_{\alpha\beta}(q^1, q^2, \ldots, q^n)\, f^\beta(t) - A_\alpha^3, \qquad f^\beta(t) = \int F^\beta(t)\, dt$$

Here we take into account that fact that $F^\beta(t)$ are scalar functions of time only, therefore the absolute integral

$$\hat{\int} F^\beta(t)\, Dt$$

is equal to the integral

$$\int F(t) dt$$

and we thus obtain

$$\partial T/\partial \dot{q}^{\alpha} = tR_\alpha(q^1, \ldots, q^n) + S_{\alpha\beta}(q^1, \ldots, q^n)\, f^\beta(t) + A_\alpha \quad (1.5)$$

$$A_\alpha = A_\alpha^1 + A_\alpha^2 + A_\alpha^3$$

where the vector A_α is so far undetermined.

In accordance with the assertions made in [1, 2], the vector A_α represents in this case the vector $\partial T / \partial \dot{q}^{\alpha} - tR_\alpha - S_{\alpha\beta} f^\beta$ at the initial instant $t = t_0$, and is translated in parallel along the trajectory into any of its points t. The problem of parallel translation of a vector is encountered in [4, 5]. In [6] a bipoint tensor is established in a specific form, and is used to compute a parallel translation from one point to another, while the author of [7] shows some properties of this tensor. Below we shall make use of the results obtained in these papers.

In the configurational space the fundamental bipoint tensor mentioned above, has the form

$$a_{\alpha k} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \left(\frac{\partial \mathbf{r}_\nu}{\partial q^k}\right)_{q^k = q^k(t_0) = q_0^k} = a_{\alpha k}(q^1, \ldots, q^n; q_0^1, \ldots, q_0^n)$$

If the initial conditions for a parallel-translatable vector A_α are known, then in accordance with [2] we obtain

$$A_\alpha = a_\alpha^{\ k}(p_k - t_0 R_k - S_{kl} f^l), \quad a_\alpha^{\ k} = a^{kl} a_{\alpha l} = a_\alpha^{\ k}(q^1, \ldots, q^n;\ q_0^1, \ldots, q_0^n)$$

where $a_\alpha^{\ k}$ denotes a mixed bipoint fundamental tensor of the configuration space V_n [6]. Therefore the first covariant integrals (1.5) of the equations of motion (1.4) are

$$\frac{\partial T}{\partial \dot{q}^{\alpha}} = tR_{\alpha}(q^1, \ldots, q^n) + S_{\alpha\beta}(q^1, \ldots, q^n) f^{\beta}(t) + \qquad (1.6)$$

$$[p_k - t_0 R_k - S_{kl} f^l(t_0)]\, a_{\alpha}^{\ k}(q^1, \ldots, q^n; q_0^1, \ldots, q_0^n)$$

Let us consider the following particular cases:

1°. When $S_{\alpha\beta} = 0$ or $f^{\beta}(t) = 0$, the generalized force is a parallel-translatable vector $Q_{\alpha} = R_{\alpha}(q^1, \ldots, q^n)$, and the relations (1.6) become

$$\partial T / \partial \dot{q}^{\alpha} = tR_{\alpha} + (p_k - t_0 R_k)\, a_{\alpha}^{\ k}$$

2°. The same form of (1.6) is retained when $R_{\alpha} = \text{const.}$

3°. When $Q_{\alpha} = R_{\alpha} = 0$, we obtain n first covariant integrals of the form

$$\partial T / \partial \dot{q}^{\alpha} = a_{\alpha}^{\ k} p_k \qquad (1.7)$$

2. The differential equations of motion (1.1) are usually called covariant ones, although their covariance is not fully preserved. We shall show this assertion, and below we shall derive the covariant differential equations of motion in a novel form which also enables us to determine the second covariant integrals.

Denoting the momentum vectors of the points in a Cartesian coordinate system by

$$\mathbf{K}_i = \{K_{3i},\ K_{3i-1},\ K_{3i-2}\} = \{K_{\nu}\} \quad (i = 1, \ldots, N;\ \nu = 1, \ldots, 3N)$$

the equations of motion of the system can be written in the form

$$dK_{\nu} / dt = F_{\nu} \qquad (2.1)$$

Comparing (2.1) with (1.2) we find that they are equivalent (2.1) $\Leftrightarrow$ (1.2). Taking into account the fact that the differential operator d/dt in the Cartesian coordinate system has a corresponding absolute differential operator D / dt in the curvilinear coordinate system, i.e. $d / dt \Leftrightarrow D / dt$ and, that $K_{\nu} \Leftrightarrow p_{\alpha}$ and $F_{\nu} \Leftrightarrow Q_{\alpha}$, we conclude that Eqs. (2.1) are covariant.

We note that the above conclusion cannot be reached when the expression dK_{ν} / dt is written in the usually adopted form as $m_i d^2\mathbf{r}_i / dt^2$. In fact, the velocity vector is expressed in terms of the derivatives dq^{α}/dt. Comparing $dx^j / dt \Leftrightarrow dq^{\alpha} / dt$ with the relations (2.1) we see that the absolute differential operator D / dt does not appear in the expressions for the velocities, i.e. the element of covariance is absent. To satisfy this requirement, we proceed from the definition [8] of the covariant position vector of the representative point in the curvilinear coordinate system

$$\rho_{\alpha} = \sum_{i=1}^{N} m_i \mathbf{r}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q^{\alpha}} \qquad (2.2)$$

the time derivative of which is

$$\frac{d\rho_{\alpha}}{dt} = a_{\alpha\beta} \dot{q}^{\beta} + \sum_{i=1}^{N} m_i \mathbf{r}_i \cdot \frac{\partial^2 \mathbf{r}_i}{\partial q^{\beta} \partial q^{\alpha}} \dot{q}^{\beta}$$

$$a_{\alpha\beta} = \sum_{i=1}^{N} m_i \frac{\partial \mathbf{r}_i}{\partial q^{\alpha}} \cdot \frac{\partial \mathbf{r}_i}{\partial q^{\beta}}$$

Since we know that $a_{\alpha\beta}$ is a metric tensor and $(\partial^2\mathbf{r}_i/\partial q^{\alpha}\partial q^{\beta}) \cdot \partial \mathbf{r}_i/\partial q^{\gamma} = \Gamma_{\alpha\beta\nu,\gamma}$ are the connectivity coefficients [9, 10], we conclude that the generalized impulse p_{α} is equal to the absolute time derivative of the vector ρ_{α}

$$p_\alpha = D\rho_\alpha/dt \tag{2.3}$$

Substituting this into (1.2), we obtain the differential equations of motion of a mechanical holonomic scleronomous system in their fully covariant and contravariant form

$$\frac{D^2\rho_\alpha}{dt^2} = Q_\alpha \Leftrightarrow \frac{D^2\rho^\beta}{dt^2} = Q^\beta \tag{2.4}$$

Comparing (2.4) with the equations $m_\nu\,(d^2x_\nu/dt^2) = F_\nu$, we see that $d^2/dt^2 \Leftrightarrow D^2/dt^2$. Comparison of (2.4) with (1.1) however does not lead to the same conclusion. This means that the Lagrange equations of second kind in their explicit form are not covariant with respect to the second time derivative. It follows that the second covariant integrals cannot be obtained from the known differential equations of motion. Incidentally, proceeding from Eqs.(2.4), we can obtain definite integrals of a system of nonlinear differential equations for a wider class of forces. For example, since $\partial T/\partial q^\alpha = D\rho_\alpha/dt$, we obtain from (1.6)

$$\hat{\int} D\rho_\alpha = \hat{\int} (tR_\alpha + S_{\alpha\beta}f^\beta)\, Dt + \hat{\int} (p_k - t_0R_k - S_{kl}f^l)\, a_\alpha{}^k Dt$$

and this yields the following covariant integrals:

$$\rho_\alpha = \frac{t^2R_\alpha}{2} + S_{\alpha\beta}\int f^\beta(t)\,dt + t\,(p_k - t_0R_k - S_{kl}f^l)\,a_\alpha{}^k + A_\alpha$$

where A_α is a parallel-translatable vector which can easily be found from the initial conditions by means of a parallel translation along the trajectory and ρ_α, R_α, $S_{\alpha\beta}$ and $a_\alpha{}^k$ are known functions of the coordinates q^α.

Equations (2.4) enable us to determine certain integrals even when velocity-dependent forces are present. For example, let the generalized forces

$$Q_\alpha = b_{\alpha\beta}\dot q^\beta = b_{\alpha\beta}\frac{D\rho^\beta}{dt}, \quad b_{\alpha\beta} = b_{\beta\alpha}(q^1, \ldots, q^n)$$

where $b_{\alpha\beta}$ is a covariantly constant vector. Then applying the absolute tensor integral and using the above method, we obtain from (2.4) n first covariant integrals

$$p_\alpha = b_{\alpha\beta}\rho^\beta + a_\alpha{}^A\,(p_A - b_{AB}\rho^B)$$

where

$$\rho^B = \rho^\beta(q_0{}^1, \ldots, q_0{}^n), \qquad b_{AB} = b_{\alpha\beta}(q_0{}^1, \ldots, q_0{}^n)$$

$$\left.\frac{D\rho_\alpha}{dt}\right|_{t=t_0} = p_A \quad (\alpha, \beta, A, B = 1, 2, \ldots, n)$$

3. Equations (2.4) also describe the motion of a rigid body. To prove this, it is sufficient to determine the value of the fundamental tensor $a_{\alpha\beta}$ and of the connectivity coefficients $\Gamma^\alpha_{\beta\gamma}$ for the coordinate system in which the motion of the rigid body is considered. Let the radius vector of the i-th point of the body be $\mathbf{r}_i = \mathbf{r}_c + \mathbf{r}'_i$, the angular velocity vector $\omega = \dot q^{\alpha'} e_{\alpha'}$ $(\alpha' = 4, 5, 6)$ and the velocity of a point of the body

$$\frac{\partial \mathbf{r}_i}{\partial q^\alpha}\dot q^\alpha = \frac{\partial \mathbf{r}_c}{\partial q^{\alpha''}}\dot q^{\alpha''} + \frac{\partial \mathbf{r}_i'}{\partial q^{\alpha'}}\dot q^{\alpha'} = \dot q^{\alpha''}\mathbf{e}_{\alpha''} + \dot q^{\alpha'}(\mathbf{e}_{\alpha'} \times \mathbf{r}_i')$$

$$(\alpha = 1, 2, \ldots, 6;\ \alpha'' = 1, 2, 3)$$

Since $\mathbf{e}_{\alpha''}$ and $\mathbf{e}_{\alpha'} \times \mathbf{r}_i$ are independent vectors, we have

$$\frac{\partial \mathbf{r}_i}{\partial q^{\alpha'}} = \frac{\partial \mathbf{r}_i'}{\partial q^{\alpha'}} = \mathbf{e}_{\alpha'} \times \mathbf{r}_i', \quad \frac{\partial \mathbf{r}_i}{\partial q^{\alpha''}} = \mathbf{e}_{\alpha''}$$

The coordinates of the tensor $a_{\alpha\beta}$ are, by definition,

$$a_{\alpha''\beta''} = \int\limits_m \frac{\partial \mathbf{r}}{\partial q^{\alpha''}} \cdot \frac{\partial \mathbf{r}}{\partial q^{\beta''}} dm = \int\limits_m \mathbf{e}_{\alpha''} \cdot \mathbf{e}_{\beta''} dm = m g_{\alpha''\beta''}$$

$$a_{\alpha'\beta'} = \int [(\mathbf{e}_{\alpha'} \cdot \mathbf{e}_{\beta'}) (\mathbf{r}' \cdot \mathbf{r}^{7}) - (\mathbf{r}' \cdot \mathbf{e}_{\alpha'}) (\mathbf{r}' \cdot \mathbf{e}_{\beta'})] \, dm =$$

$$\int\limits_m (g_{\alpha'\beta'} g_{\gamma'\delta'} - g_{\alpha'\gamma'} g_{\beta'\delta'}) \, \rho^{\gamma'} \rho^{\delta'} \, dm = I_{\alpha'\beta'}$$

Here $g_{\alpha''\beta''}$ is the metric tensor of the geometrical manifold in which the motion of a rigid body of mass m is considered, and $I_{\alpha'\beta'}$ is the inertia tensor [11]. Thus the fundamental tensor $a_{\alpha\beta}$ represents an inertial matrix in the following generalized form:

$$a_{\alpha\beta} = \begin{Vmatrix} m g_{\alpha''\beta''} & 0 \\ 0 & I_{\alpha'\beta'} \end{Vmatrix} \tag{3. 1}$$

We shall also show the validity of the relations

$$\Gamma^{\alpha}_{\beta\gamma} \frac{dq^{\gamma}}{dt} = \omega_{\beta}{}^{\alpha} \tag{3, 2}$$

where $\omega_{\beta}{}^{\alpha} = \omega^{\gamma} e_{\beta\gamma}{}^{\alpha}$ is a skew-symmetric tensor [12] of the elementary rotation. The time derivative of the vector $\mathbf{r}'$ is

$$\frac{d\mathbf{r}'}{dt} = (\rho^{\cdot\alpha'} + \rho^{\beta'} \omega^{\alpha'}_{\beta'}) \, \mathbf{e}_{\alpha'}$$

since $e_{\beta}{}^{\cdot} = \omega_{\beta}{}^{\alpha} e_{\alpha}$. On the other hand we have

$$\frac{d\mathbf{r}}{dt} = \frac{D\rho^{\alpha}}{dt} \mathbf{e}_{\alpha}, \text{ т. e. } \rho^{\cdot\alpha} + \Gamma^{\alpha}_{\beta\gamma} \rho^{\beta} q^{\cdot\gamma} = \rho^{\cdot\alpha} + \omega_{\beta}{}^{\alpha} \rho^{\beta}$$

and this yields the relations (3, 2). Consequently Eqs. (2. 4) also represent the covariant equations of motion of a rigid body. Under the conditions (3. 1) and (3. 2) these equations have the form

$$\frac{D^2\rho_{\alpha}}{dt^2} = \frac{d}{dt}\left(\frac{D\rho_{\alpha}}{dt}\right) + \omega_{\alpha}{}^{\gamma} \frac{D\rho_{\gamma}}{dt} = Q_{\alpha}$$

Taking into account the fact that the impulses $D\rho_{\alpha}/dt = p_{\alpha} = I_{\alpha\beta}\omega^{\beta}$, we obtain from these relations the well known Euler equations.

4. If the generalized forces are equal to zero, then

$$\frac{D\rho_{\alpha}}{dt} = a^{k}_{\alpha} p_k = a^{k}_{\alpha} a_{kl} q_0^{\cdot l} = a_{\alpha l} q_0^{\cdot l} \tag{4. 1}$$

This shows that, since the bipoint tensor $a_{\alpha l}$ is a covariantly constant tensor and $q_0^{\cdot l}$ are the constant initial parameters, the impulses p_{α} remain covariantly constant along the trajectory. The relations (4. 1) can also be written in a different manner (s is the arc of the trajectory)

$$\frac{D\rho_{\alpha}}{ds} s^{\cdot} = a_{\alpha k} \left(\frac{dq^k}{ds}\right)_{s=s_0} s_0^{\cdot}$$

Since here we also have $s^{\cdot} = s_0^{\cdot} = \text{const}$, it follows that:

$$\frac{D\rho_{\alpha}}{ds} = a_{\alpha k} \left(\frac{dq^k}{ds}\right)_{s=s_0}$$

and the latter relation represents the differential equations of the geodesic [6]. Using the absolute tensor integral, we obtain the geodesic equations in their final form [13]

$$\rho_{\alpha} = a_{\alpha k} q'^{k} (s - s_0) + a_{\alpha k} \rho^{k} \tag{4. 2}$$

The same result can be obtained using the Galilean transformations $\mathbf{r}_i = \mathbf{r}_{0i} + \mathbf{v}_{0i}(t - t_0)$ $(i = 1, \ldots, N)$. Scalar multiplying these equations by $m_i \partial \mathbf{r}_i / \partial q^\alpha$ and summing over i, we obtain

$$\sum_{i=1}^{N} m_i \mathbf{r}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q^\alpha} = S_1 + S_2 (t - t_0) \tag{4.3}$$

$$S_1 = \sum_{i=1}^{N} m_i \mathbf{r}_{i0} \cdot \frac{\partial \mathbf{r}_i}{\partial q^\alpha}, \quad S_2 = \sum_{i=1}^{N} m_i \mathbf{v}_{0i} \frac{\partial \mathbf{r}_i}{\partial q^\alpha}$$

Using the arguments given above we obtain, in accordance with the well known notation

$$S_1 = \sum_{\nu=1}^{3N} m_\nu x_0{}^\nu \frac{\partial x^\nu}{\partial q^\alpha} = \sum_{\nu=1}^{3N} m_\nu \left(\frac{\partial x^\nu}{\partial q^k}\right)_{t=t_0} \frac{\partial x^\nu}{\partial q^\alpha} \rho_0{}^k = a_{k\alpha} \rho_0{}^k$$

$$S_2 = \sum_{i=1}^{N} m_i \left(\frac{\partial \mathbf{r}_i}{\partial q^k}\right)_{t=t_0} \cdot \frac{\partial \mathbf{r}_i}{\partial q^\alpha} \left(\frac{dq^k}{ds} \frac{ds}{dt}\right)_{t=t_0} = a_{k\alpha} q'^k s_0^\cdot$$

Substituting these into (4.3), we obtain the covariant Galilean transformation in the form

$$\rho_\alpha = a_{\alpha k} \rho_0{}^k + s_0^\cdot a_{\alpha k} q'^k (t - t_0) \tag{4.4}$$

Since $s^\cdot = s_0^\cdot = (s - s_0)/(t - t_0)$, the equations (4.2) follow from (4.4). Therefore, when the generalized forces are absent, the representative point of a mechanical system moves along the geodesic in the successive direction. This assertion is based on the Galilean transformations and on definite integration of the system of equations of motion. It is also shown that the generalized impulse is covariantly constant during the motion of a holonomic mechanical system when the interactions of its points cancel each other out.

The above results embrace the fundamental Hertz law [14], the Newton's and Galileo's postulates and the Galilean transformations. They make possible the reformulation of the first postulate of dynamics in the following form: if the interactions between the bodies cancel out, then the impulse of the motion of the system is covariantly constant and the representative point of the system moves along the geodesic in the successive direction.

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ON A PROBLEM OF POINCARÉ

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We consider the behavior of the integral

$$I(t) = \int_0^t f(\omega_1 t, \omega_2 t)\, dt$$

as $t \to +\infty$. Here $f(\varphi_1, \varphi_2)$ is a continuous function of a two-dimensional torus $T^2\{\varphi_1\varphi_2 \bmod 1\}$, and the ratio of the frequencies ω_2 / ω_1 is irrational. The problem was first studied by Poincaré [1] and is often encountered in analytic studies of the dynamic systems.

It is well know [1, 2] that if

$$\int_0^1 \int_0^1 f(\varphi_1, \varphi_2)\, d\varphi_1\, d\varphi_2 > 0 \ (< 0)$$

then $I(t) \to +\infty\ (-\infty)$ as $t \to +\infty$. A difficulty arises when the mean spatial value of the function f is zero. Poincaré used examples to show in [1] that in this case the integral $I(t)$ can tend either to $+\infty$ or to $-\infty$ (like t^α, $0 < \alpha < 1$) and, in the most interesting case, be unbounded but able to approach its initial value (equal to zero) infinitely many times and as closely as required. A question naturally arises of determining the conditions under which the integral $I(t)$ will be recurrent (Poisson stability). The first step towards this solution consists of inspecting the discrete analog of the problem, and this helps us to establish that the recurrence takes place if the function f is twice continuously differentiable.

1. We assume that a continuous function $f(x)$ is given on the circumference $S^1\{x \bmod 1\}$. Let α be an irrational number. We construct the sum

$$S_N(\alpha, \varphi) = \sum_{i=0}^{N-1} f(i\alpha + \varphi), \quad \varphi \in S^1$$

If